Finite Element Beginnings

Platform: Windows
Requires Mathcad 3.1 or higher, 5 MB hard disk space
Available for immediate download (size 3517782 bytes) or ground shipment

This Electronic Book, by engineer and teacher David Pintur, is an introduction to the principles of the finite element method. If you use, or intend to use, existing finite element packages but want a deeper theoretical understanding of the methodology, this book is ideal. Through a variety of examples, you get a solid foundation for establishing finite element applications so you can move on to more advanced programs. And, because it's based on Mathcad's "live" math environment, every number, formula, and plot can be adapted to solve your individual problems. You can change parameters and plots and watch Mathcad recalculate answers right there in the book. Selecting interpolation functions, assembling stiffness matrices, and solving for field variables are just some of the areas covered in this Electronic Book.

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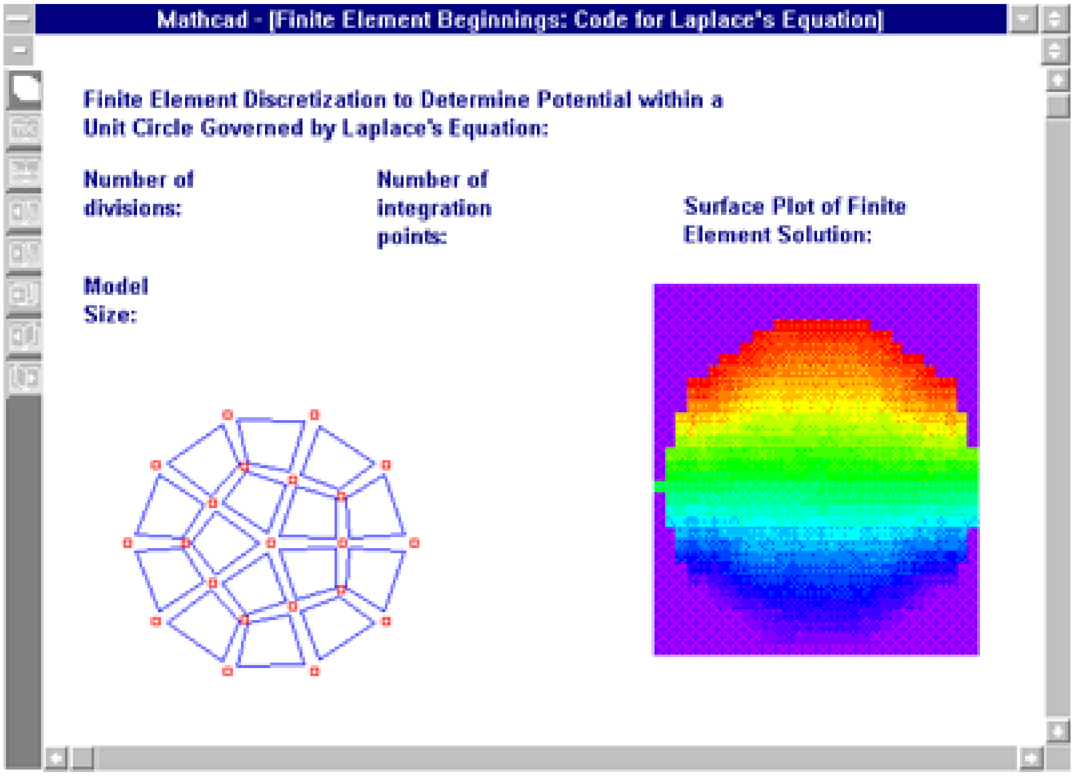

The potential within a unit circle, given by the Laplace Equation, can be determined using finite element discretization.

Topics include: Historical Perspective of the Finite Element Method, Basic Concepts of Linear Elasticity, the Principles of Minimum Potential Energy and Direct Method, Using Interpolation Concepts in One and Two Dimensions, Mapped Elements, and much more.

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This Electronic Book is unique in that it merges the actual computer implementation with the method's theoretical basis. The Mathcad environment allows the reader freedom to experiment and explore the concepts which are introduced. As the title implies, Finite Element Beginnings provides an introduction to the method which the reader can use as a springboard to more advanced issues and applications.

Introduction

Definition and Basic Concepts

The Process of Discretization

- Discrete Systems
- Continuous Systems
- Comparison to the Finite Difference Method

- Seven Basic Steps of the Finite Element Method
  - Discretizing the Continuum
  - Selecting Interpolation Functions
  - Finding Element Equations
  - Assembling the Elements
  - Applying the Boundary Conditions
  - Solving the System of Equations
  - Making Additional Computations

Brief History of the Finite Element Method

The Discrete Approach: A Physical Interpretation

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A Simple Elastic Spring

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- Step 1: Discretize the Spring System
- Step 2: Select Interpolation Functions
- Step 3: Find the Element Properties
- Step 4: Assemble the Elements
- Step 5: Apply the Boundary Conditions
- Step 6: Solve the System of Equations
- Step 7: Additional Calculations

Assembling the Elements

- An Example Finite Element Mesh
- The Assembly Algorithm
- Properties of the Assembled Stiffness Matrix

How to Treat Boundary Conditions

- The Direct Method
- The Payne and Irons Technique
- Matrix Partitioning

A Discrete Finite Element Algorithm in One Dimension

- Application to Other Discrete Systems

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- Element Stiffness Matrix in Global Coordinates
- Stiffness Derivation Using Local Coordinates

A Finite Element Algorithm for Trusses in Two Dimensions

- Truss Algorithm with Discussion
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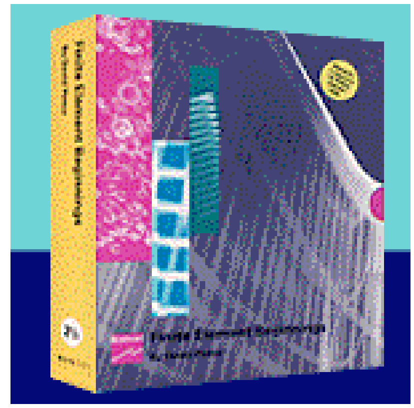

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- Plane Stress and Plane Strain

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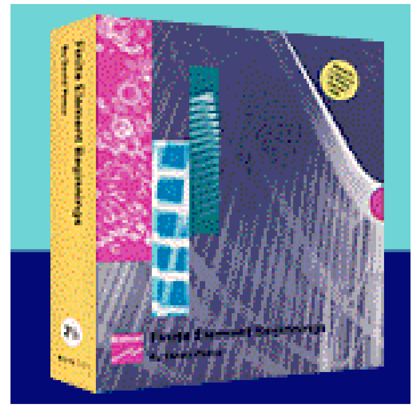

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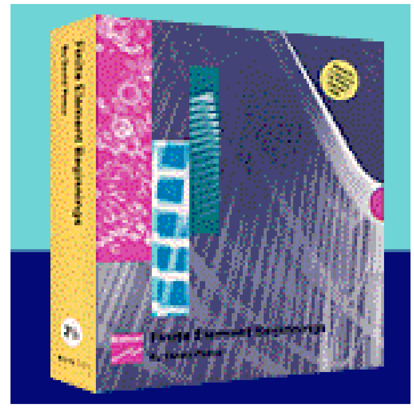

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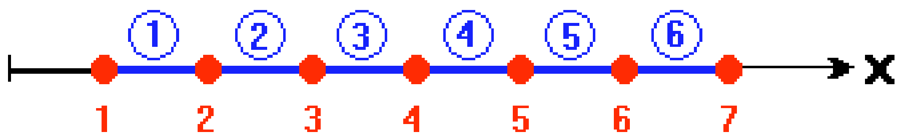
Piecewise Linear Interpolation

In the previous section, a linear variation of the field variable valid over the domain of a single element was defined. Now, a linear interpolation over a series of connected elements is shown in the figure below:

Figure 1: Example of Finite Element Mesh



$$x_{node} := (1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7)^T \qquad n_{el} := 6$$

Suppose that the exact solution to the field variable is

$$\phi(x) := 7 \cdot \sin(1.5 \cdot x) + 8$$

and the field variable at each node corresponds to the exact solution:

$$i := 1 .. \mathrm{rows}(x_{node}) \qquad \phi_{e_i} := \phi\left(x_{node_i}\right)$$

$$\phi_e^T = (\ 14.982\quad 8.988\quad 1.157\quad 6.044\quad 14.566\quad 10.885\quad 1.842\ )$$

Using the linear shape functions, N1 and N2, the field variable across the domain of all the elements can be approximated in a piecewise manner, as demonstrated in the following algorithm:

Shape functions

$$N_1(x, x_1, x_2) := \frac{x - x_2}{x_1 - x_2} \qquad N_2(x, x_1, x_2) := \frac{x_1 - x}{x_1 - x_2}$$

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Piecewise Linear Interpolation

Define x values along each element

$n := 25$ $\qquad j := 1 .. n_{el}$

$$\begin{pmatrix} x_{j,1} \\ x_{j,n+2} \end{pmatrix} := \begin{pmatrix} x_{node_j} \\ x_{node_{j+1}} \end{pmatrix} \qquad k := 2 .. n+1$$

$$\Delta_j := \frac{x_{node_{j+1}} - x_{node_j}}{(n-1)} \qquad x_{j,k} := x_{node_j} + \Delta_j \cdot (k-2)$$

Variation of field variable

$$\phi_{j,k} := N_1\left(x_{j,k}, x_{node_j}, x_{node_{j+1}}\right) \cdot \boldsymbol{\phi}_{e_j} \cdots + N_2\left(x_{j,k}, x_{node_j}, x_{node_{j+1}}\right) \cdot \boldsymbol{\phi}_{e_{j+1}} \qquad \begin{pmatrix} \phi_{j,1} \\ \phi_{j,n+2} \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$k := 1 .. n+2$

Figure 2: Plot of Field Variable over Entire Domain

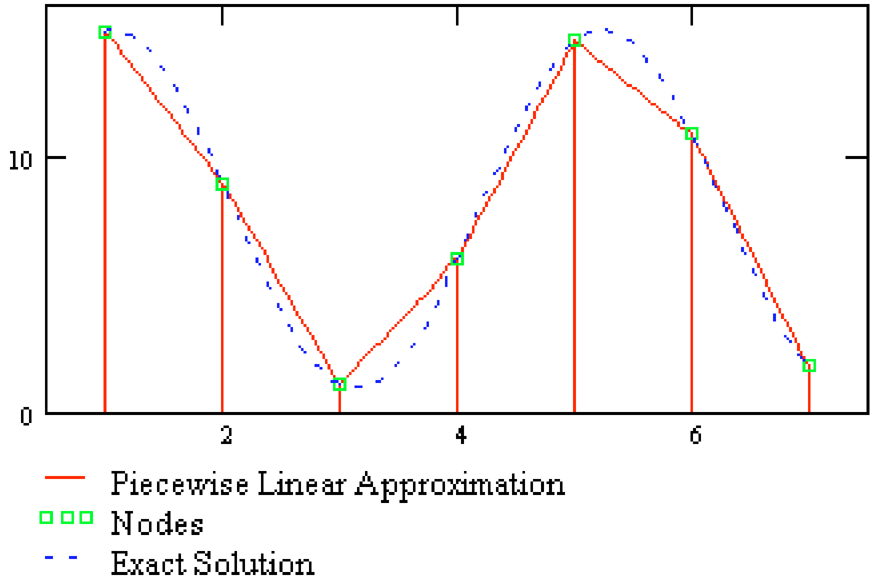

Figure 2 demonstrates a **piecewise linear interpolation** between the values of the field variable at each node. It is **piecewise** because the distribution is calculated one piece at a time, element by element, and the value of f(x) within the element is interpolated **linearly** from the nodal values.

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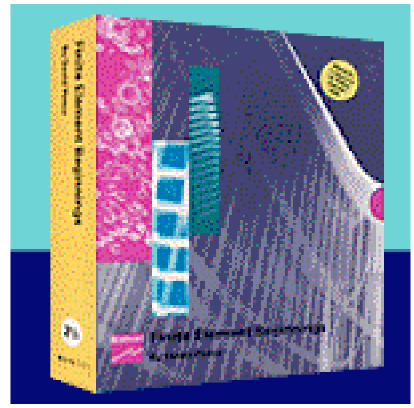

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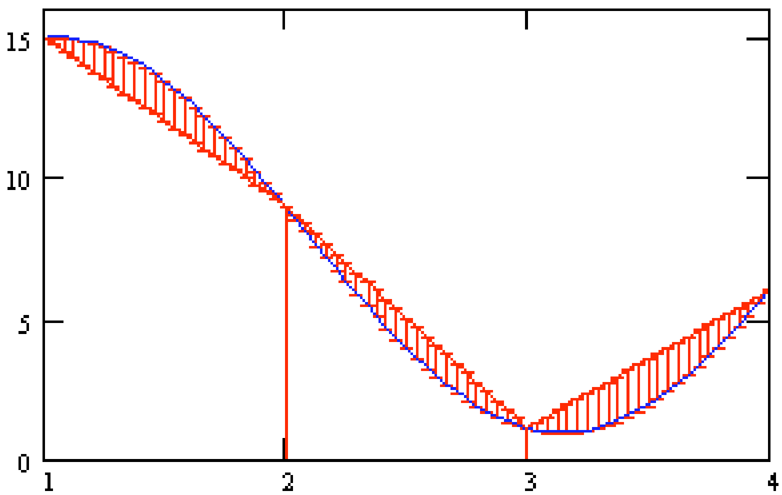
Error Analysis

The relative error, fj,k, for the first three elements is redrawn:

$j := 1 .. 3$ $\qquad$ $k := 1 .. n + 2$

Figure 3: Close-up of First Three Elements



Observe that the piecewise representation of the field variable over the physical domain of the problem is exact at the node points while it is only an approximation across each element domain. The error in the approximation is the shaded area between the exact curve and the linear approximation. To increase the accuracy of the approximation, then the following could be done:

1. Increase the number of elements over the domain.
2. Keep the same number of elements, but increase the order of the interpolation.

Increasing the Order of Interpolation

For rapidly varying functions, a linear approximation would require a large amount of elements to accurately reflect the function's gradients.

Alternatively, higher-order elements could be formulated that would also have higher-order interpolation functions and more nodes per element.

The next section investigates higher-order interpolation functions.