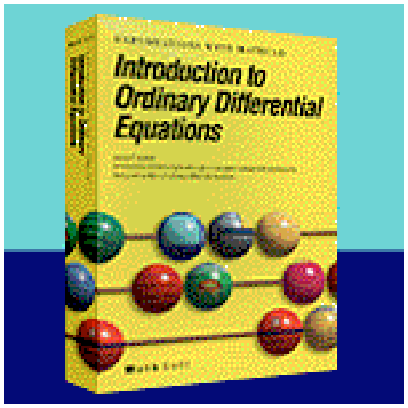
Explorations with Mathcad:

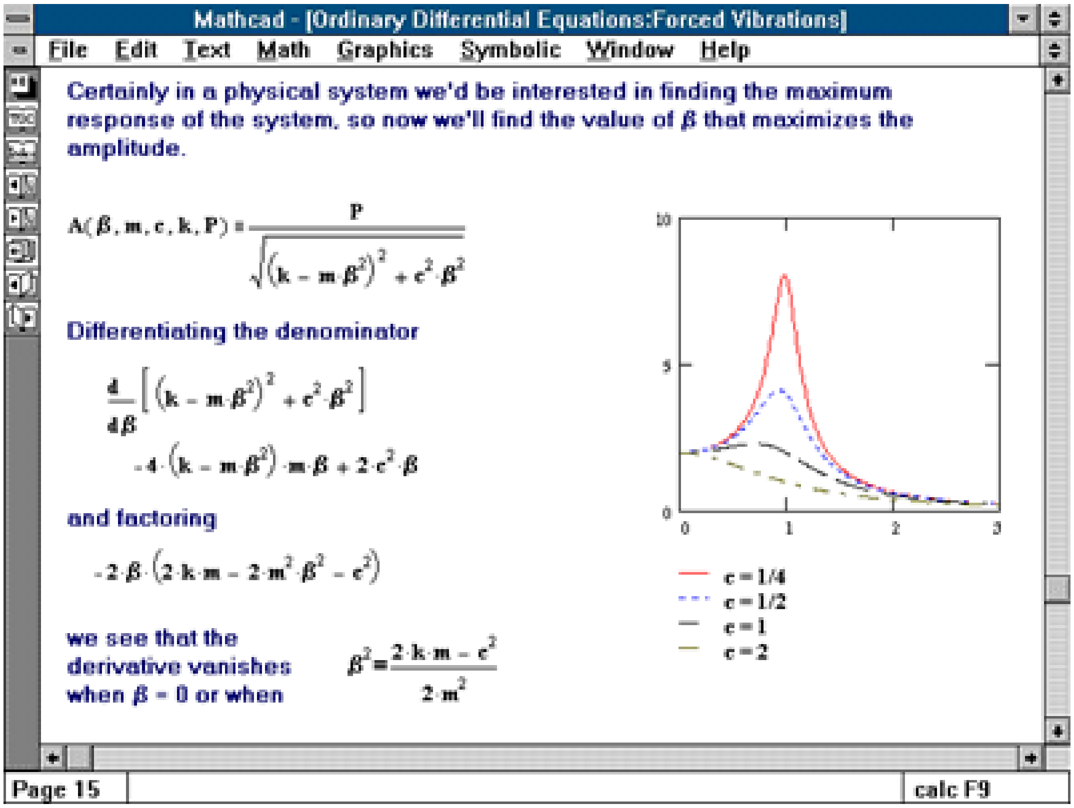
Introduction to Ordinary Differential Equations



Platform: Windows
Includes the Mathcad Engine
Available for immediate download (size 3033036 bytes) or ground shipment

This Electronic Book, authored by Lila F. Roberts, mathematics professor at Georgia Southern University, is a concise introduction to the theory of ordinary differential equations (ODEs). It covers exact methods and numerical approximations of solutions, and emphasizes graphing to illustrate concepts and streamline solution techniques. Examples and applications come to life when implemented in Mathcad. Teachers and students can experiment with different methods and see the effects of different initial conditions and step sizes.

Table of Contents | Product Sample | Back to Product List



Using differentiation to solve a "real-life" problem: optimizing the value of a parameter in order to maximize the response of a physical system.

Areas of study include: Introduction to ODEs, First Order Equations and Applications, Systems of First Order Linear ODEs, Higher Order Linear ODEs, Nonlinear Systems and Series Solutions of ODEs. Specific topics covered include Solution and Integral Curves, Homogeneous and Nonhomogeneous Systems, Undetermined Coefficients, Predator-Prey Interactions, Bessel's Equation, and much more.

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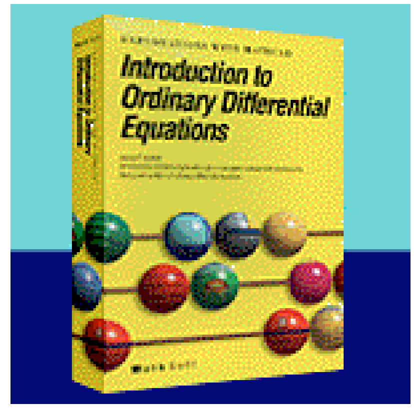


TABLE OF CONTENTS (page 1 of 3)

Product Sample

Back to Product List

Explorations with Mathcad:

Introduction to Ordinary Differential Equations

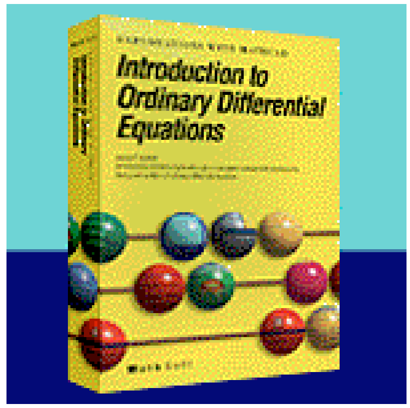


TABLE OF CONTENTS (page 2 of 3)

Product Sample

Back to Product List

Explorations with Mathcad:

Introduction to Ordinary Differential Equations

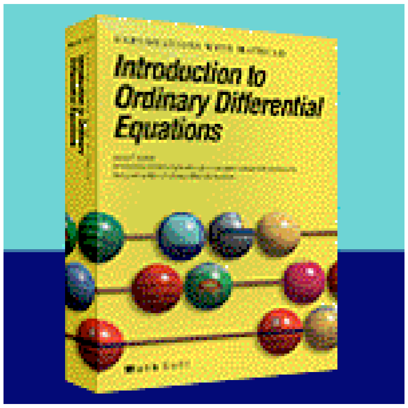
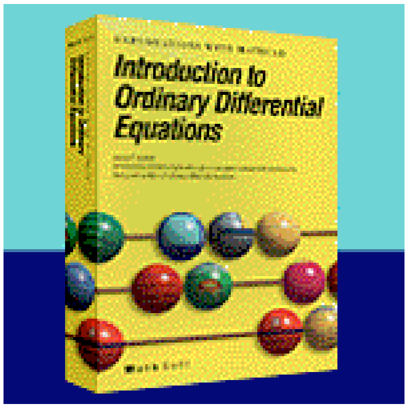


TABLE OF CONTENTS (page 3 of 3)

Explorations with Mathcad:

Introduction to Ordinary Differential Equations



Algebraically Homogeneous Differential Equations

Whenever a differential equation has an integrating factor, it is possible to change an equation that we cannot immediately solve into one which can be solved by integrating. Using an integrating factor involves only multiplication of every term in the differential equation by a function.

In this section we introduce yet another type of differential equation, many instances of which cannot be solved by methods we've discussed so far. We begin with an example.

Example 1

Consider the following differential equation: $\left(x^2 + y^2\right)\cdot dx + x\cdot y\cdot dy = 0$

This equation is not separable or exact, and an integrating factor is not immediately obvious. However, if we rewrite the equation as

$$x\cdot y\cdot\frac{d}{dx}y = -\left(x^2 + y^2\right)$$

and do some algebra

$$\frac{d}{dx}y = -\frac{x}{y} - \frac{y}{x}$$

we see that the left hand side is a function of y/x, so the differential equation can be written in the
form

$$\frac{d}{dx}y = f\left(\frac{y}{x}\right)$$

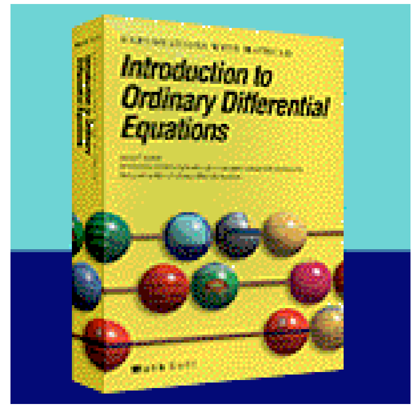
This form of the equation suggests that we might make the change in variables:

$$v = \frac{y}{x} \quad \text{or} \quad y = x\cdot v$$

Table of Contents

Back to Product List

Introduction to Ordinary Differential Equations



SAMPLE PAGE (page 2 of 10)

Then since

$$\frac{d}{dx}y = x\cdot\frac{d}{dx}v + v$$

the differential equation can be transformed to a differential equation in which v is the unknown
function.

$$x\cdot\frac{d}{dx}v + v = f(v)$$

In our example, the differential equation was

$$\frac{d}{dx}y = -\frac{x}{y} - \frac{y}{x}$$

the change of variables

$$y = x\cdot v$$

the right hand side becomes

$-\frac{x}{y} - \frac{y}{x}$ by substitution, yields $\frac{-1}{v} - v$

and the left hand side is

$$x\cdot\frac{d}{dx}v + v$$

so the differential equation becomes

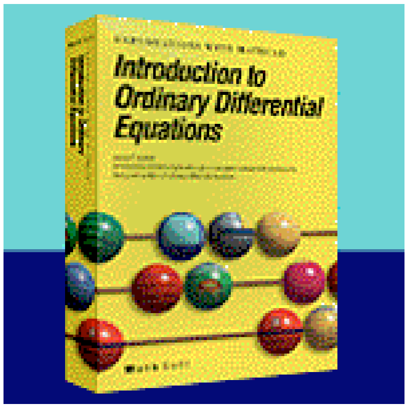
$x\cdot\left(\frac{d}{dx}v\right) = \frac{-1}{v} - 2\cdot v$ simplifies to $x\cdot\left(\frac{d}{dx}v\right) = \frac{-\left(1 + 2\cdot v^2\right)}{v}$

Table of Contents

Back to Product List

Explorations with Mathcad:

Introduction to Ordinary Differential Equations



This differential equation is separable. Separating the variables, we obtain

$$-\frac{v}{1+2\cdot v^2}\cdot dv = \frac{1}{x}\cdot dx$$

Integrating the left hand side

$-\frac{v}{1+2\cdot v^2}$ by integration, yields $\frac{-1}{4}\cdot\ln\left(1+2\cdot v^2\right)$

Integrating on the right

$\frac{1}{x}$ by integration, yields $\ln(x)$

The general solution to the differential equation, in terms of v and x is

$\frac{-1}{4}\cdot\ln\left(1+2\cdot v^2\right) = \ln(x)+C$ C is arbitrary.

This simplifies to $\left(1+2\cdot v^2\right)^{-\frac{1}{4}} = K\cdot x$

In terms of x and y, since v = y/x $\left[1+2\cdot\left(\frac{y}{x}\right)^2\right]^{-\frac{1}{4}} = K\cdot x$

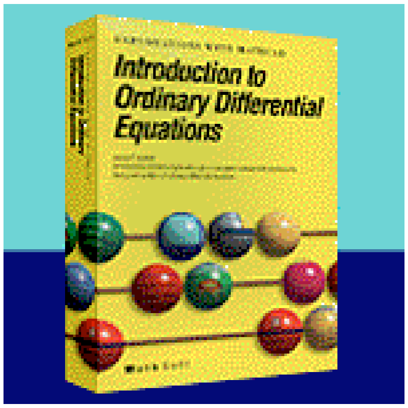
Of course, all the algebra that we've performed here depends on values of x and y that make the expressions meaningful.

Table of Contents

Back to Product List

Explorations with Mathcad:

Introduction to Ordinary Differential Equations



When can we use this procedure? There is a definite class of differential equations for which the type of transformation we used in the previous example is fruitful. Before we define that class of differential equations, we'll look at some functions that are called ***algebraically homogeneous functions***.

Algebraically Homogeneous Functions

A function f = f(x,y) is said to be ***algebraically homogeneous of order n*** whenever it has the property that

$f(t\cdot x, t\cdot y) = t^n \cdot f(x, y)$ for any nonzero number t.

Example 1:

Given $f(x, y) = x^2 + y^2$

Then $f(t\cdot x, t\cdot y) = (t\cdot x)^2 + (t\cdot y)^2 = t^2\cdot\left(x^2 + y^2\right)$

The function $f(x, y) = x^2 + y^2$

is algebraically homogeneous of degree 2.

Example 2:

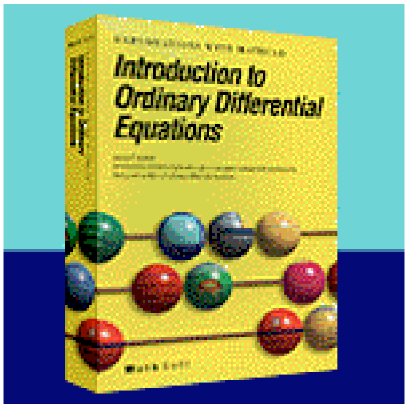
Given the function $f(x, y) = x^2 + y^2 + 3\cdot x$

Then $f(t\cdot x, t\cdot y) = (t\cdot x)^2 + (t\cdot y)^2 + 3\cdot t\cdot x \neq t^n\cdot f(x, y)$

The given function is not algebraically homogeneous.

Explorations with Mathcad:

Introduction to Ordinary Differential Equations



Algebraically Homogeneous Differential Equations

A differential equation of the form

$$M(x,y)\cdot dx + N(x,y)\cdot dy = 0$$

is said to be algebraically homogeneous provided that M(x,y) and N(x,y) are each algebraically homogeneous functions of order n.

Any algebraically homogeneous differential equation can be solved by rewriting the equation in the form

$$\frac{d}{dx}y = f\left(\frac{y}{x}\right)$$

and making the transformation of the dependent variable

$$y = x\cdot v$$

The transformed differential equation will be

$$x\cdot\left(\frac{d}{dx}v\right) + v = f(v)$$

which is separable.

Note: Any homogeneous differential equation can be written in the form

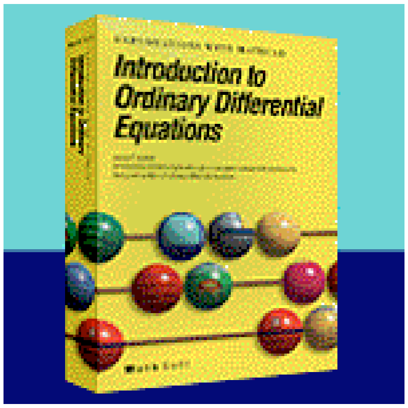
$$\frac{d}{dx}y = f\left(\frac{y}{x}\right)$$

by expressing M(x,y) and N(x,y) in the form

$$M(x,y) = x^n\cdot M\left(1,\frac{y}{x}\right) \qquad \text{and} \qquad N(x,y) = x^n\cdot N\left(1,\frac{y}{x}\right)$$

Table of Contents

Back to Product List



We can do this since both M and N are algebraically homogeneous. So the differential equation can be expressed as

$$\frac{d}{dx}y=\frac{-M(x,y)}{N(x,y)}=\frac{-x^n\cdot M\left(1,\frac{y}{x}\right)}{x^n\cdot N\left(1,\frac{y}{x}\right)}=\frac{M\left(1,\frac{y}{x}\right)}{N\left(1,\frac{y}{x}\right)}$$

or equivalently as

$$\frac{d}{dx}y=f\left(\frac{y}{x}\right)$$

Example 3:

Solve the initial value problem:

$$x\cdot\left(\frac{d}{dx}y\right)=y+x\cdot e^{\frac{y}{x}} \qquad y(1)=0$$

Solution: Assuming that

$$x\neq 0$$

the differential equation is equivalent to

$$\frac{d}{dx}y=\frac{y}{x}+e^{\frac{y}{x}}$$

Let $y=x\cdot v$

then an equivalent form for the differential equation is

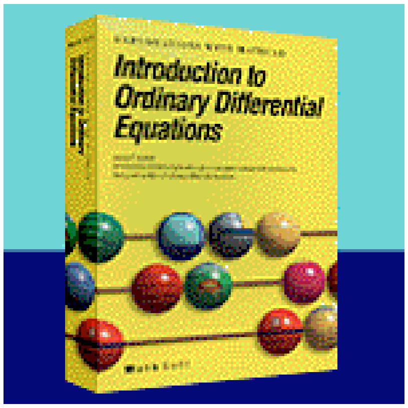
$$x\cdot\frac{d}{dx}v+v=v+e^{v}$$

Table of Contents

Back to Product List

Explorations with Mathcad:

Introduction to Ordinary Differential Equations



Since $y(1)=0$

and $y=x\cdot v$

then the initial condition in terms of v is $v(1)=0$

The differential equation is separable. The differential form is $e^{-v}\cdot dv=\frac{1}{x}\cdot dx$

The integral form for the initial value problem is

$\int_0^v e^{-u}\,du=\int_1^x \frac{1}{u}\,du$ has solution(s) $-\ln(1-\ln(x))$

for $\ln(x)<1$ *The argument of the natural logarithm function must be positive!*

Remember, however, that the integral

$$\int^x \frac{1}{u}\,du$$

actually evaluates to $\ln(|x|)$

so we could write the result of integration as $-\ln(1-\ln(|x|))$

Since $y=x\cdot v$

and $v=\frac{y}{x}$

the solution in terms of y is $y(x):=-x\cdot\ln(1-\ln(x))$

for $0<x<e$

Table of Contents

Back to Product List

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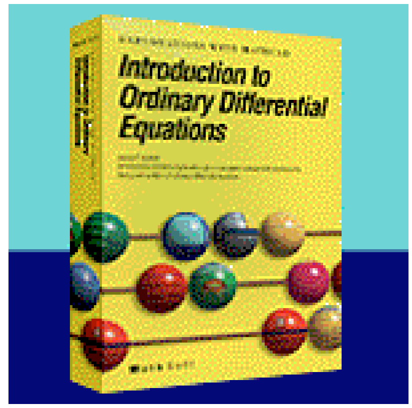
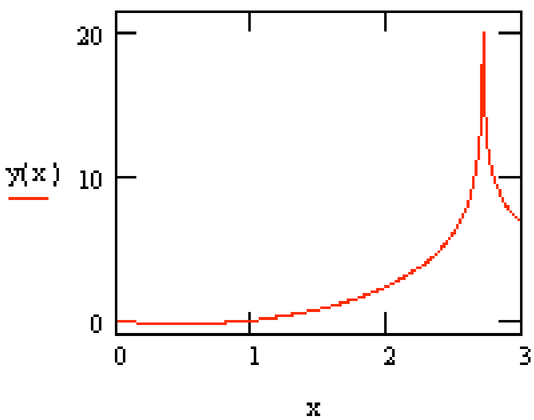
Introduction to Ordinary Differential Equations



Suppose we forget to think about allowable values for x. Let's look at a plot to discover what happens. Let's make a plot of the function over, say [0,3]:

$x := 0, 0.01 .. 3$



Looks pretty weird. What happens at x = e?

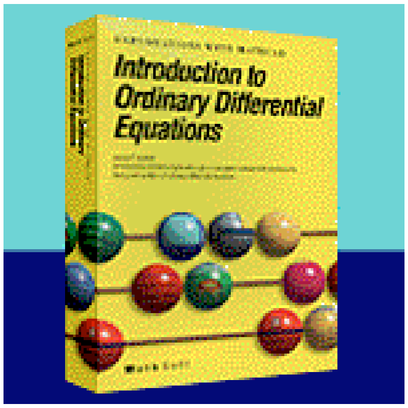
$y(e) =$

domain error

What made this happen? Look at the function definition -- is there a problem when x = e?

Let's look at what happens at x = 0.	$y(x) := -x \cdot \ln(1 - \ln(x))$
Mathcad tells us that	$y(0) = 0$
If we evaluate directly, we see that we get	$0 \cdot \ln(1 - \ln(0)) = 0$
which of course has a domain error. To see what's going on, let's take the limit as x ->0+	$-x \cdot \ln(1 - \ln(x))$
has the indeterminate form as x -> 0+	$0 \cdot \infty$

Introduction to Ordinary Differential Equations



Disregarding the negative sign for now, write in the form

$$\frac{\ln\left(1+\ln\left(\frac{1}{x}\right)\right)}{\frac{1}{x}}$$ which is indeterminate of the form

$$\frac{\infty}{\infty}$$ as x -> 0+

Use L'Hospital's rule. Differentiate the numerator and the denominator. We can let Mathcad do the
differentiation

$$\frac{d}{dx}\left(\ln\left(1+\ln\left(\frac{1}{x}\right)\right)\right)$$ yields $$\frac{-1}{(x\cdot(1-\ln(x)))}$$

which can be written $$-\frac{1}{x\cdot\left(1+\ln\left(\frac{1}{x}\right)\right)}$$

The denominator differentiates to $$-\frac{1}{x^2}$$

So now we need to evaluate the limit

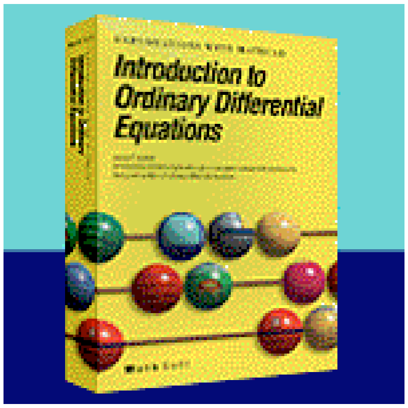
$$\frac{-\frac{1}{x\cdot\left(1+\ln\left(\frac{1}{x}\right)\right)}}{-\frac{1}{x^2}}$$ simplifies to $$\frac{-x}{(-1+\ln(x))}$$

Table of Contents

Back to Product List

Explorations with Mathcad:

Introduction to Ordinary Differential Equations



Now if we take the limit as x ->0+,

Numerator--> 0

Denominator--> $-\infty$

This tells us that as x ->0+, the y(x) --> 0. The function has a removable discontinuity at x = 0. What Mathcad does is to remove the discontinuity by defining y(0)= 0.

Let's verify that the original differential equation is satisfied for several values of x by looking at the difference:

$TOL := 10^{-6}$ $\qquad x := 1, 1.5 .. 5$

$x\cdot\frac{d}{dx}y(x) - y(x) - x\cdot\exp\left(\frac{y(x)}{x}\right)$
$-2.22\cdot10^{-15}$
$4.885\cdot10^{-15}$
$-1.767\cdot10^{-13}$
$2.32\cdot10^{-10}$
$1.009\cdot10^{-11} + 1.605i\cdot10^{-10}$
$-2.695\cdot10^{-11} - 7.843i\cdot10^{-11}$
$1.474\cdot10^{-13} + 3.423i\cdot10^{-13}$
$1.297\cdot10^{-13} + 7.925i\cdot10^{-14}$
$-1.99\cdot10^{-13} - 3.809i\cdot10^{-13}$

We expect the results to be close to zero, which they are. Note that for x > 3 we get complex results coming from the evaluation of the natural logarithms -- the function y still satisfies the differential equation nicely even though its value is now complex!