

Finite Element Beginnings

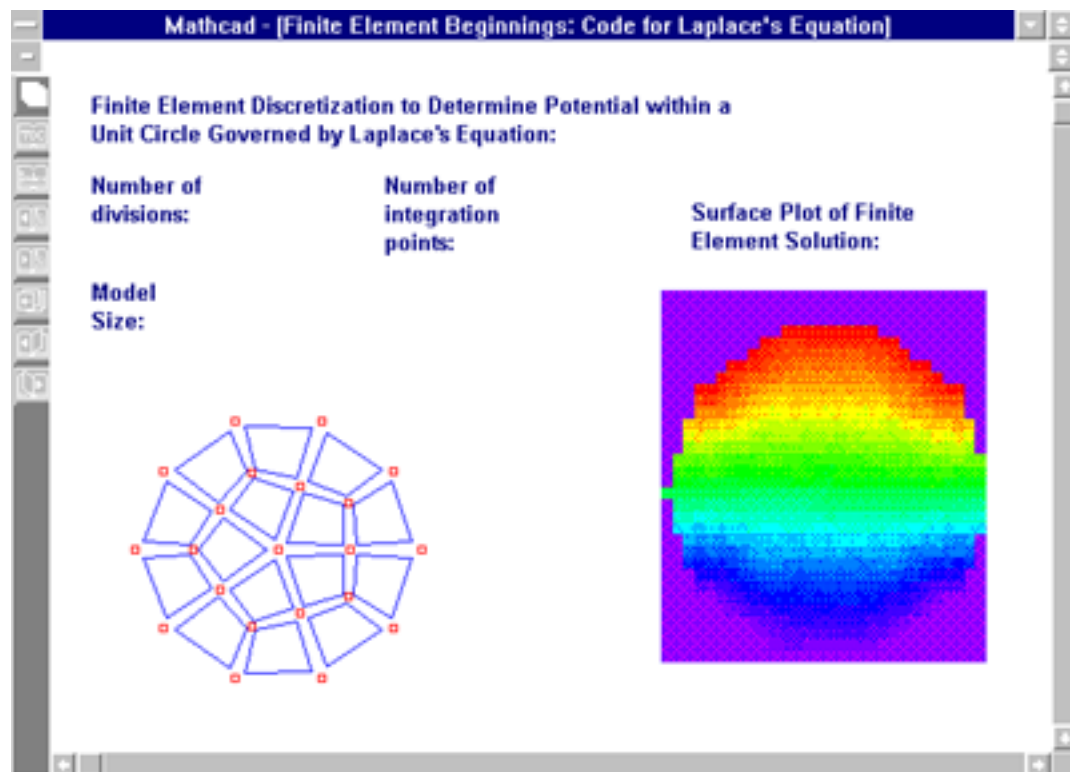
Platform: Windows

Requires Mathcad 3.1 or higher, 5 MB hard disk space

Available for immediate download (size 3517782 bytes) or ground shipment



This Electronic Book, by engineer and teacher David Pintur, is an introduction to the principles of the finite element method. If you use, or intend to use, existing finite element packages but want a deeper theoretical understanding of the methodology, this book is ideal. Through a variety of examples, you get a solid foundation for establishing finite element applications so you can move on to more advanced programs. And, because it's based on Mathcad's "live" math environment, every number, formula, and plot can be adapted to solve your individual problems. You can change parameters and plots and watch Mathcad recalculate answers right there in the book. Selecting interpolation functions, assembling stiffness matrices, and solving for field variables are just some of the areas covered in this Electronic Book.

[Table of Contents](#)[Product Sample](#)[Back to Product List](#)

The potential within a unit circle, given by the Laplace Equation, can be determined using finite element discretization.

Topics include: Historical Perspective of the Finite Element Method, Basic Concepts of Linear Elasticity, the Principles of Minimum Potential Energy and Direct Method, Using Interpolation Concepts in One and Two Dimensions, Mapped Elements, and much more.

Finite Element Beginnings



TABLE OF CONTENTS (page 1 of 6)

This Electronic Book is unique in that it merges the actual computer implementation with the method's theoretical basis. The Mathcad environment allows the reader freedom to experiment and explore the concepts which are introduced. As the title implies, Finite Element Beginnings provides an introduction to the method which the reader can use as a springboard to more advanced issues and applications.

Introduction

Definition and Basic Concepts

The Process of Discretization

Discrete Systems

Continuous Systems

Comparison to the Finite Difference Method

Seven Basic Steps of the Finite Element Method

Discretizing the Continuum

Selecting Interpolation Functions

Finding Element Equations

Assembling the Elements

Applying the Boundary Conditions

Solving the System of Equations

Making Additional Computations

Brief History of the Finite Element Method

The Discrete Approach: A Physical Interpretation

Introduction

A Simple Elastic Spring

[Product Sample](#)

[Back to Product List](#)

Finite Element Beginnings

TABLE OF CONTENTS (page 2 of 6)

A System of Springs

- Step 1: Discretize the Spring System
- Step 2: Select Interpolation Functions
- Step 3: Find the Element Properties
- Step 4: Assemble the Elements
- Step 5: Apply the Boundary Conditions
- Step 6: Solve the System of Equations
- Step 7: Additional Calculations

Assembling the Elements

- An Example Finite Element Mesh
- The Assembly Algorithm
- Properties of the Assembled Stiffness Matrix

How to Treat Boundary Conditions

- The Direct Method
- The Payne and Irons Technique
- Matrix Partitioning

A Discrete Finite Element Algorithm in One Dimension

- Application to Other Discrete Systems

Truss Analysis

- Element Stiffness Matrix in Global Coordinates
- Stiffness Derivation Using Local Coordinates

A Finite Element Algorithm for Trusses in Two Dimensions

- Truss Algorithm with Discussion
- Truss Algorithm without Discussion



[Product Sample](#)

[Back to Product List](#)

Finite Element Beginnings

TABLE OF CONTENTS (page 3 of 6)

Introduction to Finite Elements of Elastic Continua

Introduction

Continuity of Elements in a Continuum

Basic Concepts in Three Dimensional Linear Elasticity

The Displacement Field

Strain Components

Stress Components

Constitutive Laws

The Principle of Minimum Potential Energy

Plane Stress and Plane Strain

A Triangular Element in Plane Stress

The Direct Method for a Triangular Element

Interpolation of Displacement

Strain-Displacement Equation

Stress-Strain Relationship

Equivalent Forces for a Stress Field

The Stiffness Matrix

a) Summary of the Direct Method

The Energy Method for Elastic Elements

The Stiffness Matrix

How to Treat Surface Traction

Final Remarks

Comparison of the Direct and Energy Methods for Plane Stress

A Finite Element Code for Plane Strain

Plane Stress Code With Discussion

Plane Stress Code Without Discussion



[Product Sample](#)

[Back to Product List](#)

Finite Element Beginnings

TABLE OF CONTENTS (page 4 of 6)

Element Interpolation and Shape Functions

Introduction

The Essence of the Finite Element Method

Linear Interpolation in One Dimension

Piecewise Linear Interpolation

Higher-Order Polynomials in One Dimension

Quadratic Interpolation in One Dimension

Piecewise Quadratic Interpolation

Generalization to Higher Orders

Derivatives of Shape Functions

Linear Interpolation and Differentiation

Quadratic Interpolation and Differentiation

Continuity Requirements

Polynomials in Two Dimensions

A Linear Triangular Element

A Four Node Rectangular Element

A Specialized Rectangular Element

Shape Functions Using Normalized Coordinates

1-D Lagrangian Shape Functions

2-D Lagrangian Shape Functions

2-D Serendipity Shape Functions

Final Remarks

Mapped Elements

Introduction



[Product Sample](#)

[Back to Product List](#)

Finite Element Beginnings

TABLE OF CONTENTS (page 5 of 6)

Mapping in One Dimension

Differentiation and Integration

a) Newton-Cotes Quadrature

b) Gauss Quadrature

c) Summary

Mapping in Two Dimensions

Evaluation of Element Equations

Transformation of Derivatives

The Area Integral and Numerical Integration

a) Integration of Mapped Quadratic Elements

Integration Along Element Boundaries

Shape Functions Along Element Boundaries

Reduction to One Dimension on Boundaries

Evaluating a Distributed Edge Load

Finite Element Code Using Isoparametric Plane Stress Elements

Linear Isoparametric Plane Stress Elements

Quadratic Isoparametric Plane Stress Elements

The Method of Weighted Residuals

Introduction

Overview of Residual Methods

Problem Definition

Approximate Solution Using Trial Functions



[Product Sample](#)

[Back to Product List](#)

Finite Element Beginnings

TABLE OF CONTENTS (page 6 of 6)

a) Point Collocation

b) Subdomain Collocation

c) Galerkin's Method

Comparison of the Three Methods

Applying Galerkin's Method to Finite Elements

One Dimension: Integration by Parts

Finite Element Code in One Dimension

Two Dimensions: Green's Theorem

Finite Element Applications

Laplace's Equation in a Circular Disk

a) Linear Finite Element Code of Laplace's Equation

Laplace's Equation in a Rectangular Region

a) Quadratic Finite Element Code of Laplace's Equation

Concluding Remarks

References

Index



[Product Sample](#)

[Back to Product List](#)

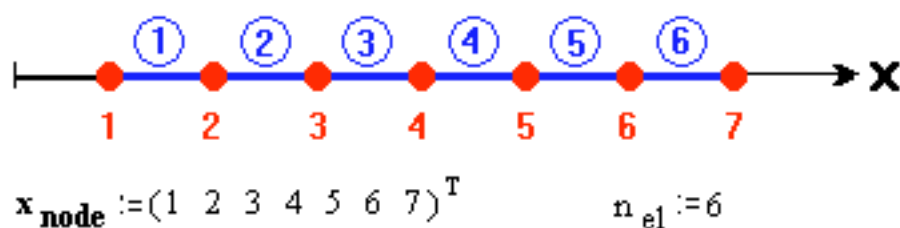
Finite Element Beginnings

SAMPLE PAGE (page 1 of 3)

Piecewise Linear Interpolation

In the previous section, a linear variation of the field variable valid over the domain of a single element was defined. Now, a linear interpolation over a series of connected elements is shown in the figure below:

Figure 1: Example of Finite Element Mesh



Suppose that the exact solution to the field variable is

$$\phi(x) := 7 \cdot \sin(1.5 \cdot x) + 8$$

and the field variable at each node corresponds to the exact solution:

$$i := 1 \dots \text{rows}(\mathbf{x}_{\text{node}}) \quad \phi_{\mathbf{e}_i} := \phi(\mathbf{x}_{\text{node}_i})$$

$$\phi_{\mathbf{e}}^T = (14.982 \quad 8.988 \quad 1.157 \quad 6.044 \quad 14.566 \quad 10.885 \quad 1.842)$$

Using the linear shape functions, N1 and N2, the field variable across the domain of all the elements can be approximated in a piecewise manner, as demonstrated in the following algorithm:

Shape functions

$$N_1(x, x_1, x_2) := \frac{x - x_2}{x_1 - x_2} \quad N_2(x, x_1, x_2) := \frac{x_1 - x}{x_1 - x_2}$$



[Table of Contents](#)

[Back to Product List](#)

Finite Element Beginnings

SAMPLE PAGE (page 2 of 3)

Piecewise Linear Interpolation

Define x values along each element

$$n := 25$$

$$j := 1 \dots n_{el}$$

$$\begin{pmatrix} x_{j,1} \\ x_{j,n+2} \end{pmatrix} := \begin{pmatrix} x_{node_j} \\ x_{node_{j+1}} \end{pmatrix}$$

$$k := 2 \dots n+1$$

$$\Delta_j := \frac{x_{node_{j+1}} - x_{node_j}}{(n-1)}$$

$$x_{j,k} := x_{node_j} + \Delta_j \cdot (k-2)$$

Variation of field variable

$$\begin{aligned} \phi_{j,k} := & N_1(x_{j,k}, x_{node_j}, x_{node_{j+1}}) \cdot \phi_j + \dots \\ & + N_2(x_{j,k}, x_{node_j}, x_{node_{j+1}}) \cdot \phi_{j+1} \end{aligned} \quad \begin{pmatrix} \phi_{j,1} \\ \phi_{j,n+2} \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$k := 1 \dots n+2$$

Figure 2: Plot of Field Variable over Entire Domain

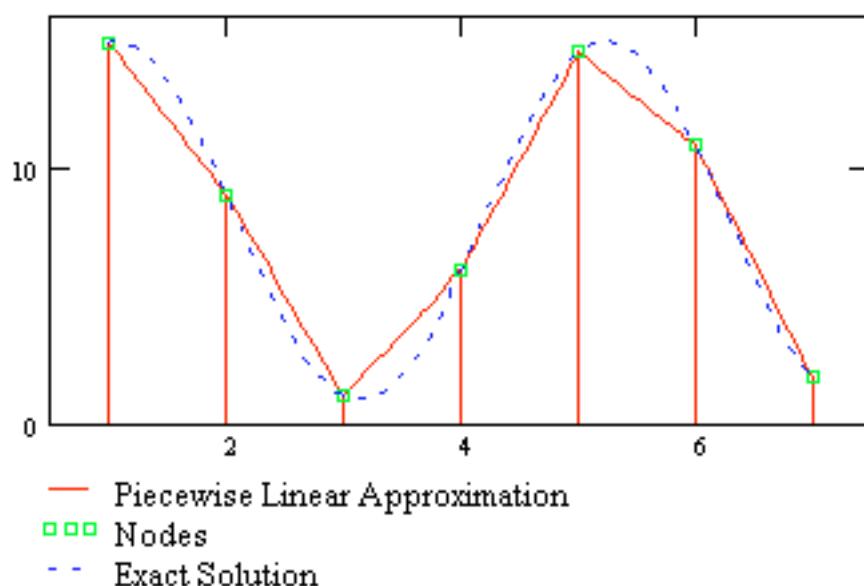


Figure 2 demonstrates a **piecewise linear interpolation** between the values of the field variable at each node. It is **piecewise** because the distribution is calculated one piece at a time, element by element, and the value of $f(x)$ within the element is interpolated **linearly** from the nodal values.



[Table of Contents](#)

[Back to Product List](#)

Finite Element Beginnings

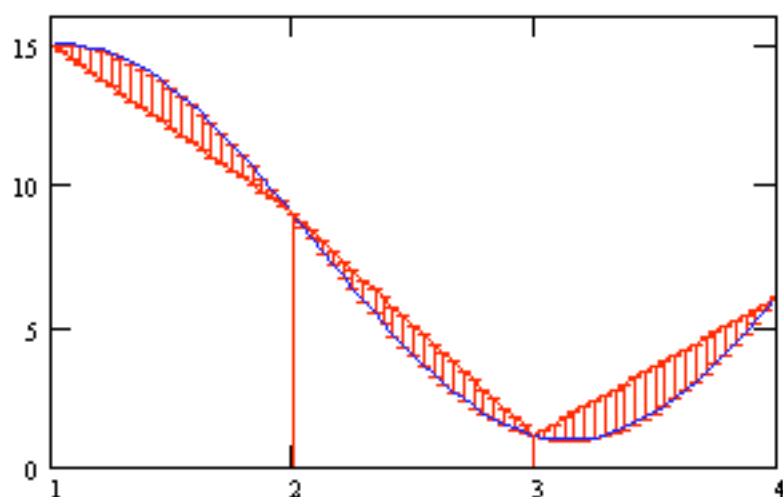
SAMPLE PAGE (page 3 of 3)

Error Analysis

The relative error, $f_{j,k}$, for the first three elements is redrawn:

$$j := 1..3 \quad k := 1..n+2$$

Figure 3: Close-up of First Three Elements



Observe that the piecewise representation of the field variable over the physical domain of the problem is exact at the node points while it is only an approximation across each element domain. The error in the approximation is the shaded area between the exact curve and the linear approximation. To increase the accuracy of the approximation, then the following could be done:

1. Increase the number of elements over the domain.
2. Keep the same number of elements, but increase the order of the interpolation.

Increasing the Order of Interpolation

For rapidly varying functions, a linear approximation would require a large amount of elements to accurately reflect the function's gradients.

Alternatively, higher-order elements could be formulated that would also have higher-order interpolation functions and more nodes per element.

The next section investigates higher-order interpolation functions.



[Table of Contents](#)

[Back to Product List](#)